Math 208 Cheatsheet

Row operations

- multiply row by (non-zero) scalar
- add/subtract multiple of any row to another
- swap any row

Echelon Form

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

Def:
- all zero rows at bottom
- all pivots to right of pivots above

the variables associated w/ pivot columns are "leading" variables

" " w/ non-pivot columns are "free" variables

ex:
$$\begin{array}{c} x_1\ x_2\ x_3\ x_4\ x_5 \\ \left[\begin{array}{ccccc|c} * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

x_2, x_5 are free variables

Reduced Row Echelon Form

- in REF
- all pivots are 1
- only zeros above pivots
- Allows you to directly read off solutions

Solutions to Systems

- any row $(0 \cdots 0 \mid a)$ w/ $a \neq 0$ indicates no solutions (inconsistent)
- exactly one solution iff system is consistent, and there are <u>no</u> free variables (i.e. pivot in every col besides rightmost in Aug)
- ∞ solutions if free variables (i.e. at least one col w/ no pivot)
- any lin sys w/ more variables than eqns cannot have only 1 soln
- homogeneous lin sys always has at least one solution (zero solution)

Matrix-Vector multiplication

- Can only multiply if inner dim match, e.g. $\mathbb{R}^{n \times m} \times \mathbb{R}^{m \times 1}$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} a \\ c \\ e \end{bmatrix} + x_2 \begin{bmatrix} b \\ d \\ f \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \\ ex_1 + fx_2 \end{bmatrix}$$

Linear Transformations

$T: \mathbb{R}^m \to \mathbb{R}^n$ – T is a transformation from <u>Domain</u> $\mathbb{R}^m$ to <u>codomain</u> $\mathbb{R}^n$

<u>Range</u> is not necessarily $\mathbb{R}^n$, but instead a subset of $\mathbb{R}^n$ (equiv, image)

$\mathrm{Im}(T) = \{T(v) : v \in \mathbb{R}^m\}$

$\mathrm{Im}(\bar{x}) = T(\bar{x})$

<u>Def</u>: A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is a lin trans iff $\forall \bar{u}, \bar{v} \in \mathbb{R}^m$, $\forall c \in \mathbb{R}$

(1) $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$

(2) $T(c\bar{u}) = cT(\bar{u})$

one-to-one:

$T(\bar{u}) = T(\bar{v}) \to \bar{u} = \bar{v}$

$\mathrm{Ker}(T) = \{0\} \iff T$ is one-to-one

onto:

Range (T) = Codomain (T), i.e.

$T: \mathbb{R}^n \to \mathbb{R}^m$, $\forall \bar{y} \in \mathbb{R}^m$, $\exists x \in \mathbb{R}^n$ s.t. $T(\bar{x}) = \bar{y}$

Linear combinations

is $\bar{V}$ a lin. comb. of $\bar{u}_1, \dots, \bar{u}_m$? i.e. can we find $c_1, \dots, c_m$ s.t. $c_1\bar{u}_1 + \cdots + c_m\bar{u}_m = \bar{V}$? } equiv to is $\bar{V} \in \mathrm{span}\{\bar{u}_1, \dots, \bar{u}_m\}$?

just solve for such a $\bar{c} \in \mathbb{R}^m$ via G.E.

$$\begin{bmatrix} | & & | \\ \bar{u}_1 & \cdots & \bar{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} | & & | & v_1 \\ \bar{u}_1 & \cdots & \bar{u}_m & \vdots \\ | & & | & v_n \end{array}\right] \to$$ perform G.E., if found to be inconsistent, <u>NO</u> L.C. exists.

Span

what is the set of vectors that are lin comb. of $\bar{u}_1, \bar{u}_2$, i.e. what are the possible $\bar{b}$s for

$$\begin{bmatrix} | & | \\ \bar{u}_1 & \bar{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\left[\begin{array}{cc|c} | & | & | \\ \bar{u}_1 & \bar{u}_2 & \bar{b} \\ | & | & | \end{array}\right] \to$$ Row reduce, then find constraints on $\bar{b}$, i.e. to make system consistent

<u>ex:</u> for 2 vectors $\bar{u}_1, \bar{u}_2 \in \mathbb{R}^3$, might get
$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & b_2 \\ 0 & 0 & b_3 - 2b_1 + 2b_2 \end{array}\right)$$
to be consistent, $b_3 - 2b_1 + 2b_2 = 0$, which is the plane that $\bar{u}_1, \bar{u}_2$ span!

The following are equiv:

(1) $\bar{u}_1, \dots, \bar{u}_m$ span $\mathbb{R}^n$

(2) $x_1\bar{u}_1 + x_2\bar{u}_2 + \cdots + x_m\bar{u}_m = \bar{b}$ has a solution $\forall \bar{b} \in \mathbb{R}^n$

(3) $\begin{bmatrix} | & & | \\ \bar{u}_1 & \cdots & \bar{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \bar{b}$ has sol $\forall \bar{b} \in \mathbb{R}^n$

!!! (4) $A = \begin{bmatrix} | & & | \\ \bar{u}_1 & \cdots & \bar{u}_m \\ | & & | \end{bmatrix}$, the REF(A) has pivot in <u>EVERY</u> row

Linear Independence

<u>test</u>: form homo col matrix $[V \mid 0]$, RR, if only $\bar{x} = 0$ trivial case, LI

Def L.D.: A set of vectors $S = \{v_1, \dots, v_n\}$ w/ $v_i \in \mathbb{R}^n$ is L.D. if $\exists \bar{c} \in \mathbb{R}^m$, $\bar{c} \neq \bar{0}$ st. $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} \bar{c} = \bar{0}$

$c_1 v_1 + \cdots + c_m v_n = \bar{0}$

Def L.I.: A set of vectors $S = \{v_1, \dots, v_m\}$ s.t. $v_i \in \mathbb{R}^n$ is L.I. if $\nexists \bar{c} \in \mathbb{R}^m$ s.t. $\bar{c} \neq 0 \wedge c_1v_1 + \cdots + c_nv_n = \bar{0}$

- A set of one vector is L.I. iff it isn't the zero vector
- any set containing zero vector is LD.

<u>Theorem</u>: $\bar{v}_1, \dots, \bar{v}_m \in \mathbb{R}^n$ are LD. iff some $\bar{v}_i \in \mathrm{span}\{\bar{v}_1, \dots, \bar{v}_m\} \setminus \{v_i\}$

The following are equiv:

(1) $\bar{u}_1, \dots, \bar{u}_m$ are L.I.

(2) $x_1\bar{u}_1 + \cdots + x_m\bar{u}_m = 0$ has only trivial sol $\bar{x} = \bar{0}$

(3) $\begin{bmatrix} | & & | \\ \bar{u}_1 & \cdots & \bar{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ has a <u>unique</u> sol (only trivial)

(4) the REF of $A = \begin{bmatrix} | & & | \\ \bar{u}_1 & \cdots & \bar{u}_m \\ | & & | \end{bmatrix}$ has a pivot in <u>every</u> column

Unifying theorem

$\bar{u}_1, \dots, \bar{u}_m$ span $\mathbb{R}^n \iff$ A has pivot in all rows

$\bar{u}_1, \dots, \bar{u}_m$ L.I. $\iff$ A has pivot in all cols

if $m = n$, L.I. $\iff$ span since pivot in every row $\iff$ pivot in every col

if $\bar{u}_1, \dots, \bar{u}_m$ are L.I.

$A\vec{x} = \vec{b}$ has at most one solution for every $\vec{b}$

since REF(A) has pivot in every col $\to$ no free variables

trivia

- if $m < n$, can $\bar{u}_1, \dots, \bar{u}_m \in \mathbb{R}^n$ span $\mathbb{R}^n$? <u>NO</u>
- if ~~m~~ $m \geq n$, can $\bar{u}_1, \dots, \bar{u}_m \in \mathbb{R}^n$ be <u>guaranteed</u> to span $\mathbb{R}^n$? <u>NO</u>

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Vector Spaces

def: Set closed under (+) and scalar (*) that contains $\bar{0}$

Fact: the span of any set of vectors forms a space

basis: given a space S, a basis for S is a L.I. (and thus minimal) set of vectors that span S.

To find a basis:

- create matrix A w/ col vectors
- RREF
- non-zero rows are basis for Row space
- columns of A that end w/ pivots form basis for Col space.

to extend into basis, just keep adding standard basis vectors

Dimension: cardinality of basis

ex: basis for $\mathbb{R}^2$ is $\{e_1, e_2\}$, so $\dim(\mathbb{R}^2) = 2$

note: all bases of a space have the same size, i.e. the dim

Some vector spaces:

- $\ker(T) = \{\bar{x} : T(\bar{x}) = 0\}$ (or null(T))
  i.e. solutions $\bar{x}$ s.t. $A\bar{x} = 0$

think about where the basis vectors go!

MM is equiv to function composition

- Range (A) is span of columns
- Range (T) is everything in codomain reachable from domain

- All subspaces are either the kernel or range of some transformation

- for any plane
  $$ax + by + cz = 0$$
  this plane is kernel of
  $$\begin{bmatrix} a & b & c \end{bmatrix}$$

- Set of all quadratics is a vector space

- to prove vector space, can show it is kernel or range of some T

Row span doesn't change when converting to RREF

$$\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A))$$

Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$x_1 = ae + bg$

$x_2 = af + bh$

$x_3 = ce + dg$

$x_4 = cf + dh$

Only defined for $\mathbb{R}^{a \times n} \times \mathbb{R}^{n \times b}$

i.e. inner dim needs to match

$$A(B + C) = AB + AC$$

$$A(BC) = (AB)C$$

Matrix Inverse

$A^{-1}A = I$ (left inverse)

$AA^{-1} = I$ (right inverse)

true inverse only def for $n \times n$

- must be RR to I
- if $\det(A) = 0$, not inv

Solve:

- form $[A|I]$
- RREF
- If $[I|B]$, $B = A^{-1}$,
- else impossible

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Rank Nullity Theorem

rank: dim col or row space

nullity: dim of null (ker) space, or # of non pivot in RR form

Theorem

$A \in \mathbb{R}^{m \times n}$,

$$n = \mathrm{rank}(A) + \mathrm{nullity}(A)$$

change of bases

what is $\bar{v}_B$ w.r.t. the standard basis?

Let B be matrix of cols of basis

$B = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$ $\bar{v}_B = \begin{bmatrix} a \\ b \end{bmatrix}$

$\bar{v} = a\begin{bmatrix} c \\ e \end{bmatrix} + b\begin{bmatrix} d \\ f \end{bmatrix}$

what is $\bar{v}$ wrt basis B? ($\bar{v}_B$)

$\bar{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $B = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$ ← change of basis matrix

$\begin{bmatrix} a \\ b \end{bmatrix} = x_1\begin{bmatrix} c \\ e \end{bmatrix} + x_2\begin{bmatrix} d \\ f \end{bmatrix}$

$= \begin{bmatrix} c & d \\ e & f \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\bar{v}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\bar{v}_B = B^{-1}\bar{v}$

in general:

if $B = \{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$,

then $U = [v_1 \cdots v_n]$ is the C.O.B. matrix that changes elements $\bar{v}_B$ to $\bar{v}$

O.T.O.H. U^{-1} changes $\bar{v}$ to $\bar{v}_B$

$B_1 \to B_2$

let $U_1 : B_1 \to$ std

$U_2 : B_2 \to$ std

then $U_2^{-1}U_1$ converts from $B_1 \to B_2$

O.T.O.H. $B_2 \to B_1$ is $U_1^{-1}U_2$

L.T. in terms of non-std basis

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ 90° rot in terms of std b.

let U be C.O.B. matrix $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ for basis B

to find equiv transformation to A but in basis of B

do $U^{-1}AU\bar{v}$, i.e.

(3) (2) (1)

(1) convert $B \to$ std

(2) apply normal A in std

(3) convert from std $\to B$

determinants

Scaling of area/vol/etc. of space from std basis in image of a L.T.

$\mathbb{R}^2$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(A) = ad - bc$

general def

Let $A \in \mathbb{R}^{n \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Let A_{ij} be $(n-1) \times (n-1)$ matrix deleting i^{th} row and j^{th} col

cofactor $C_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$

$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

(or) $= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

} cofactor expansion